MEAN AND VARIANCE OF BINOMIAL DISTRIBUTION

You know that if $P(x_1), P(x_2), \ldots, P(x_n)$ are the probabilities corresponding to values $x_1, x_2, \ldots, x_n$ of a random variable, its *mean* or *average* or *expected value* or *expectation* is defined as

$$E(X) = \mu = \sum_{i=1}^{n} x_i P(x_i)$$

In case of binomial distribution, we have

$$P(r) = {}^nC_r p^r q^{n-r} \text{ for } r = 0, 1, \ldots, n$$

$$\mu = \sum_{r=0}^{n} rP(r) = \sum_{r=0}^{n} r\, {}^nC_r p^r q^{n-r}$$

$$= 0 + 1.n.pq^{n-1} + 2.\frac{n(n-1)}{2!}p^2q^{n-2} + \ldots + np^n$$

$$= np\left[q^{n-1} + (n-1)\,pq^{n-2} + \frac{(n-1)(n-2)}{2!}p^2q^{n-3} + \ldots + p^{n-1}\right]$$

$$= np\,(q+p)^{n-1} = np \qquad (\because q + p = 1)$$

Hence, the mean of the binomial distribution is np.

Now let us derive the formulae for variance and standard deviation of a binomial distribution.

$$\sigma^2 = \sum_{r=0}^{n} r^2P(r) - \mu^2 = \sum_{r=0}^{n} r^2\, {}^nC_r p^r q^{n-r} - \mu^2$$

$$= \sum_{r=0}^{n} \{r + r(r-1)\}\, {}^nC_r p^r q^{n-r} - \mu^2 \qquad (\because r^2 = r + r(r-1))$$

$$= \sum_{r=0}^{n} r\, {}^nC_r p^r q^{n-r} + \sum_{r=0}^{n} r(r-1)\, {}^nC_r p^r q^{n-r} - \mu^2$$

$$= np + \sum_{r=2}^{n} r(r-1)\, {}^nC_r p^r q^{n-r} - \mu^2$$

($\because$ Contribution of $r = 0$ and $r = 1$ in second summation is 0)

Now $\sum_{r=2}^{n} r(r-1)\, {}^nC_r p^r q^{n-r} = \sum_{r=2}^{n} r(r-1) . \frac{n(n-1)...(n-r+1)}{r!} p^r q^{n-r}$

$$= \sum_{r=2}^{n} \frac{n(n-1)(n-2)...(n-r+1)}{(r-2)!} p^r q^{n-r}$$

$$= n(n-1)p^2 \sum_{r=2}^{n} {}^{n-2}C_{r-2}\, p^{r-2} q^{n-r}$$

$$= n(n-1)p^2 (q+p)^{n-2} = n(n-1)p^2 \qquad (\because q + p = 1)$$

Thus, $\sigma^2 = np + n(n-1)p^2 - (np)^2 = np - np^2$

$= np\,(1-p) = npq$

Hence, the variance of the binomial distribution is $\sigma^2 = npq$.

It follows that the standard deviation is $\sigma = \sqrt{npq}$.

For example, tossing a fair coin is a binomial event with p = probability of heads = $\frac{1}{2}$ and $q = \frac{1}{2}$. If a coin is tossed 400 times, we expect that the coin will show heads in $np = (400)\frac{1}{2} = 200$ cases.

Standard deviation = $\sqrt{npq} = \sqrt{400.\frac{1}{2}.\frac{1}{2}} = \sqrt{100} = 10.$

Remark

Mean and standard deviation are used to test whether a coin or a die etc. is biased or unbiased. If a fairly large number of trials are made and actual observation is made, we say that the coin or die is

(i) *unbiased* if value lies in Mean ± 2σ

(ii) *perhaps biased* if it lies between two and three sigmas

(iii) *definitely biased* if it lies outside three sigmas.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the mean of the Binomial distribution* $B\left(4, \frac{1}{3}\right)$.

Solution. The parameters of the given binomial distribution are 4 and $\frac{1}{3}$ *i.e.* $n = 4$ and $p = \frac{1}{3}$.

$\therefore$ The mean of the binomial distribution = $np = 4 \times \frac{1}{3} = \frac{4}{3}$.

Example 2. *An unbiased coin is tossed 4 times. Find the probability distribution of the number of heads obtained. Hence, find the mean and the variance of the distribution.*

Solution. When a fair coin is tossed, probability of getting a head = $p = \frac{1}{2}$.

So, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$.

Here number of trials = number of times the coin is tossed = 4.

Thus, we get a binomial distribution with $p = \frac{1}{2}$, $q = \frac{1}{2}$ and $n = 4$.

$\therefore$ $P(r \text{ successes}) = {}^4C_r\, p^r q^{4-r} = {}^4C_r \cdot \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{4-r} = {}^4C_r \left(\frac{1}{2}\right)^4.$

Let X denote the number of heads obtained, then X can take values 0, 1, 2, 3, 4.

$P(0) = {}^4C_0 \left(\frac{1}{2}\right)^4 = \frac{1}{16}$, $\quad P(1) = {}^4C_1 \left(\frac{1}{2}\right)^4 = \frac{4}{16}$,

$P(2) = {}^4C_2 \left(\frac{1}{2}\right)^4 = \frac{6}{16}$, $\quad P(3) = {}^4C_3 \left(\frac{1}{2}\right)^4 = \frac{4}{16}$ and

$P(4) = {}^4C_4 \left(\frac{1}{2}\right)^4 = \frac{1}{16}$.

$\therefore$ Probability distribution is $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \frac{1}{16} & \frac{4}{16} & \frac{6}{16} & \frac{4}{16} & \frac{1}{16} \end{pmatrix}$

Mean of number of heads obtained $= \Sigma p_i x_i$

$$= \frac{1}{16} \times 0 + \frac{4}{16} \times 1 + \frac{6}{16} \times 2 + \frac{4}{16} \times 3 + \frac{1}{16} \times 4$$

$$= \frac{0+4+12+12+4}{16} = \frac{32}{16} = 2.$$

Now $\quad \Sigma p_i x_i^2 = \frac{1}{16} \times 0^2 + \frac{4}{16} \times 1^2 + \frac{6}{16} \times 2^2 + \frac{4}{16} \times 3^2 + \frac{1}{16} \times 4^2$

$$= \frac{0+4+24+36+16}{16} = \frac{80}{16} = 5.$$

$\therefore$ $\quad$ Variance $= \Sigma p_i x_i^2 - \mu^2 = 5^2 - 2^2 = 1.$

Example 3. *A coin is biased so that the head is 3 times likely to occur as tail. If the coin is tossed twice, then find the probability distribution of the number of tails. Hence, find the mean of the number of tails.*

Solution. The coin is so biased that the head is 3 times likely to occur as tail. If p is the probability of getting a tail, then probability of getting a head = $3p$,

so $\quad p + 3p = 1 \Rightarrow 4p = 1 \Rightarrow p = \frac{1}{4}$,

so $\quad q = 1 - \frac{1}{4} = \frac{3}{4}$.

The coin is tossed twice, events are independent, therefore, it is a problem of binomial distribution with $p = \frac{1}{4}$, $q = \frac{3}{4}$ and $n = 2$.

If X denotes the number of tails, then X can take values 0, 1, 2.

$P(0) = {}^2C_0\, q^2 = 1 \cdot \left(\frac{3}{4}\right)^2 = \frac{9}{16}$,

$P(1) = {}^2C_1\, pq = 2 \cdot \frac{1}{4} \cdot \frac{3}{4} = \frac{6}{16}$ and

$P(2) = {}^2C_2\, p^2 = 1 \cdot \left(\frac{1}{4}\right)^2 = \frac{1}{16}$.

$\therefore$ $\quad$ The probability distribution of the number of tails is $\begin{pmatrix} 0 & 1 & 2 \\ \frac{9}{16} & \frac{6}{16} & \frac{1}{16} \end{pmatrix}$.

Mean $= \Sigma p_i x_i = \frac{9}{16} \times 0 + \frac{6}{16} \times 1 + \frac{1}{16} \times 2 = \frac{8}{16} = \frac{1}{2}$.

Example 4. *Find the probability distribution of the number of sixes in three throws of a die. Also find the mean and variance of the distribution.*

Solution. When a die is thrown, probability of getting a six $= p = \frac{1}{6}$,

so $q = 1 - p = 1 - \frac{1}{6} = \frac{5}{6}$. Here $n = 3$.

Thus, we have a binomial distribution with $p = \frac{1}{6}, q = \frac{5}{6}$ and $n = 3$.

If X denotes the number of sixes obtained, then X can take values 0, 1, 2, 3.

$$P(0) = {}^3C_0\, q^3 = \left(\frac{5}{6}\right)^3 = \frac{125}{216},$$

$$P(1) = {}^3C_1\, pq^2 = 3.\frac{1}{6}.\left(\frac{5}{6}\right)^2 = \frac{75}{216},$$

$$P(2) = {}^3C_2\, p^2q = 3.\left(\frac{1}{6}\right)^2.\frac{5}{6} = \frac{15}{216} \text{ and}$$

$$P(3) = {}^3C_3\, p^3 = \left(\frac{1}{6}\right)^3 = \frac{1}{216}.$$

$\therefore$ Required probability distribution is $\begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{125}{216} & \frac{75}{216} & \frac{15}{216} & \frac{1}{216} \end{pmatrix}$.

$\therefore$ $$\text{Mean} = \mu = \Sigma p_i x_i = \frac{125}{216}\times 0 + \frac{75}{216}\times 1 + \frac{15}{216}\times 2 + \frac{1}{216}\times 3 = \frac{108}{216} = \frac{1}{2}.$$

$$\Sigma p_i x_i^2 = \frac{125}{216}\times 0^2 + \frac{75}{216}\times 1^2 + \frac{15}{216}\times 2^2 + \frac{1}{216}\times 3^2$$

$$= \frac{75}{216} + \frac{60}{216} + \frac{9}{216} = \frac{144}{216} = \frac{2}{3}.$$

$\therefore$ $$\text{Variance} = \Sigma p_i x_i^2 - \mu^2 = \frac{2}{3} - \left(\frac{1}{2}\right)^2 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}.$$

Alternatively

Here, we have a binomial distribution with $p = \frac{1}{6}, q = \frac{5}{6}$ and $n = 3$.

To find the mean and variance of the distribution, we have:

$$\text{mean} = np = 3 \times \frac{1}{6} = \frac{1}{2} \text{ and}$$

$$\text{variance} = npq = 3 \times \frac{1}{6} \times \frac{5}{6} = \frac{5}{12}.$$

Example 5. *Find the probability distribution of the number of doublets in four throws of a pair of dice. Hence, find the mean and variance of this distribution.*

Solution. When a pair of dice is thrown, the sample space has 36 equally likely outcomes, out of which six outcomes are doublets—(1, 1), (2, 2), (3, 3), (4, 4), (5, 5) and (6, 6).

If p = probability of getting a doublet, then $p = \frac{6}{36} = \frac{1}{6}$,

so $q = 1 - \frac{1}{6} = \frac{5}{6}$. Here $n = 4$.

Thus, we have a binomial distribution with $p = \frac{1}{6}, q = \frac{5}{6}$ and $n = 4$.

If X denotes the number of doublets obtained, then X takes the values 0, 1, 2, 3, 4.

$$P(0) = {}^4C_0\, q^4 = \left(\frac{5}{6}\right)^4 = \frac{625}{1296},$$

$$P(1) = {}^4C_1\, pq^3 = 4 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^3 = \frac{500}{1296},$$

$$P(2) = {}^4C_2\, p^2q^2 = 6\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 = \frac{150}{1296},$$

$$P(3) = {}^4C_3\, p^3q = 4\left(\frac{1}{6}\right)^3 \frac{5}{6} = \frac{20}{1296} \text{ and}$$

$$P(4) = {}^4C_4\, p^4 = \left(\frac{1}{6}\right)^4 = \frac{1}{1296}.$$

$\therefore$ The required probability distribution is $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \frac{625}{1296} & \frac{500}{1296} & \frac{150}{1296} & \frac{20}{1296} & \frac{1}{1296} \end{pmatrix}$.

$$\text{Mean} = \mu = \Sigma\, p_i x_i = \frac{1}{1296}(625 \times 0 + 500 \times 1 + 150 \times 2 + 20 \times 3 + 1 \times 4)$$

$$= \frac{1}{1296} \times 864 = \frac{2}{3}.$$

$$\text{Now } \Sigma\, p_i x_i^2 = \frac{1}{1296}(625 \times 0^2 + 500 \times 1^2 + 150 \times 2^2 + 20 \times 3^2 + 1 \times 4^2)$$

$$= \frac{1}{1296}(500 + 600 + 180 + 16) = \frac{1296}{1296} = 1.$$

$$\therefore \quad \text{Variance} = \Sigma\, p_i x_i^2 - \mu^2 = 1 - \left(\frac{2}{3}\right)^2 = 1 - \frac{4}{9} = \frac{5}{9}.$$

Example 6. *From a lot of 15 bulbs which include 5 defective bulbs, a sample of 4 bulbs is drawn one by one with replacement. Find the probability distribution of the number of defective bulbs. Hence, find the mean of the distribution.*

Solution. Total number of bulbs in the lot = 15,

number of defective bulbs = 5.

Let the probability of a defective bulb drawn be p, then $p = \frac{5}{15} = \frac{1}{3}$,

$\therefore$ the probability of a good bulb $= q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$.

As 4 bulbs are drawn one by one with replacement, the events are independent therefore it is a problem of binomial distribution with $p = \frac{1}{3}$, $q = \frac{2}{3}$ and $n = 4$.

Let random variable X denote the number of defective bulbs drawn, then X can take values 0, 1, 2, 3, 4.

$$P(X = 0) = {}^4C_0\, q^4 = 1 \times \left(\frac{2}{3}\right)^4 = \frac{16}{81},$$

$$P(X = 1) = {}^4C_1\, pq^3 = 4 \times \frac{1}{3} \times \left(\frac{2}{3}\right)^3 = \frac{32}{81},$$

$$P(X = 2) = {}^4C_2\, p^2q^2 = 6 \times \left(\frac{1}{3}\right)^2 \times \left(\frac{2}{3}\right)^2 = \frac{24}{81},$$

$$P(X = 3) = {}^4C_3\,p^3q = 4 \times \left(\frac{1}{3}\right)^3 \times \frac{2}{3} = \frac{8}{81} \text{ and}$$

$$P(X = 4) = {}^4C_4\,p^4 = 1 \times \left(\frac{1}{3}\right)^4 = \frac{1}{81}.$$

$\therefore$ The probability distribution of X is $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \frac{16}{81} & \frac{32}{81} & \frac{24}{81} & \frac{8}{81} & \frac{1}{81} \end{pmatrix}$.

$$\text{Mean} = \Sigma\, p_i x_i = \frac{1}{81}(16 \times 0 + 32 \times 1 + 24 \times 2 + 8 \times 3 + 1 \times 4)$$

$$= \frac{1}{81}(0 + 32 + 48 + 24 + 4) = \frac{108}{81} = \frac{4}{3}.$$

Example 7. *Three cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of spades. Hence, find the mean of the distribution.*

Solution. Probability of drawing a spade from a pack of 52 cards $= p = \frac{13}{52} = \frac{1}{4}$,

so, $q = 1 - p = 1 - \frac{1}{4} = \frac{3}{4}$.

As the cards are drawn successively with replacement, events are independent, therefore, it is a problem of binomial distribution with $p = \frac{1}{4}$, $q = \frac{3}{4}$ and $n = 3$.

Let random variable X denote the number of spades, so X can take values 0, 1, 2, 3.

$$P(X = 0) = {}^3C_0\,q^3 = 1\,.\left(\frac{3}{4}\right)^3 = \frac{27}{64}$$

$$P(X = 1) = {}^3C_1\,p\,.\,q^2 = 3\,.\,\frac{1}{4}\left(\frac{3}{4}\right)^2 = \frac{27}{64}$$

$$P(X = 2) = {}^3C_2\,p^2q^1 = 3\left(\frac{1}{4}\right)^2.\frac{3}{4} = \frac{9}{64}$$

$$P(X = 3) = {}^3C_3\,p^3 = 1\,.\left(\frac{1}{4}\right)^3 = \frac{1}{64}$$

$\therefore$ Required probability distribution is $\begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{27}{64} & \frac{27}{64} & \frac{9}{64} & \frac{1}{64} \end{pmatrix}$.

$$\text{Mean} = \Sigma\, p_i x_i = \frac{1}{64}(27 \times 0 + 27 \times 1 + 9 \times 2 + 1 \times 3)$$

$$= \frac{1}{64}(0 + 27 + 18 + 3) = \frac{48}{64} = \frac{3}{4}.$$

Example 8. *The mean and the variance of a binomial distribution are 4 and 2 respectively. Find the probability of atleast 6 successes.*

Solution. According to given, we have

mean $= 4 \Rightarrow np = 4$ ...(*i*)

and variance $= 2 \Rightarrow npq = 2$ ...(*ii*)

From (*i*) and (*ii*), we get

$$4.q = 2 \Rightarrow q = \frac{1}{2}, \text{ so } p = 1 - q = 1 - \frac{1}{2} = \frac{1}{2}.$$

From (*i*), we get $n \cdot \frac{1}{2} = 4 \Rightarrow n = 8.$

If r denotes the number of successes, then

$$P(r) = {}^nC_r p^r q^{n-r} = {}^8C_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{8-r} = {}^8C_r \left(\frac{1}{2}\right)^8.$$

$\therefore$ The probability of atleast 6 successes

$$= P(6) + P(7) + P(8)$$

$$= {}^8C_6 \left(\frac{1}{2}\right)^8 + {}^8C_7 \left(\frac{1}{2}\right)^8 + {}^8C_8 \left(\frac{1}{2}\right)^8$$

$$= \left(\frac{1}{2}\right)^8 ({}^8C_2 + {}^8C_1 + {}^8C_0)$$

$$= \frac{1}{256}(28 + 8 + 1) = \frac{37}{256}.$$

Example 9. *Determine the binomial distribution whose mean is 9 and whose standard deviation is $\frac{3}{2}$. Find the probability of obtaining atmost one success.*

Solution. Let the binomial distribution be $(q + p)^n$.

Given that mean $= np = 9$ and standard deviation $= \sqrt{npq} = \frac{3}{2}$

$$\Rightarrow \quad \sqrt{9q} = \frac{3}{2} \Rightarrow 3\sqrt{q} = \frac{3}{2} \Rightarrow q = \frac{1}{4}.$$

$$\therefore \quad p = 1 - q = 1 - \frac{1}{4} = \frac{3}{4}$$

and $\quad np = 9 \Rightarrow n \times \frac{3}{4} = 9 \Rightarrow n = 12.$

Hence, the binomial distribution is $(q + p)^n$ *i.e.* $\left(\frac{1}{4} + \frac{3}{4}\right)^{12}$.

$$P(\text{atmost one success}) = P(X \le 1) = P(0) + P(1)$$

$$= {}^{12}C_0 p^0 q^{12} + {}^{12}C_1 p^1 q^{11}$$

$$= 1.\left(\frac{1}{4}\right)^{12} + 12.\frac{3}{4}\left(\frac{1}{4}\right)^{11} = 37\left(\frac{1}{4}\right)^{12}.$$

Example 10. *A pair of dice is thrown 200 times. If getting a sum of 9 is considered a success, then find the mean and the variance of the number of successes.*

Solution. When a pair of dice is thrown, the sample space has 36 equally likely outcomes.

Outcomes favourable to sum of 9 are (5, 4), (4, 5), (6, 3), (3, 6) only.

$$\therefore \quad p = \text{probability (sum is 9)} = \frac{4}{36} = \frac{1}{9},$$

so $\quad q = 1 - \frac{1}{9} = \frac{8}{9}, n = 200.$

Hence, mean $= np = 200\left(\frac{1}{9}\right) = \frac{200}{9}$ and

variance $= npq = 200\left(\frac{1}{9}\right)\left(\frac{8}{9}\right) = \frac{1600}{81}.$

Example 11. *In a binomial distribution, the sum of its mean and variance is 1.8. Find the probability of two successes if the event was conducted 5 times.*

Solution. Here $np + npq = 1.8$ and $n = 5$

$\Rightarrow \quad 5p(1+q) = 1.8 \Rightarrow (1-q)(1+q) = 0.36 \quad \Rightarrow 1 - q^2 = 0.36$

$\Rightarrow \quad q^2 = 0.64 \Rightarrow q = 0.8$ $\qquad (\because 0 \le q \le 1)$

$\therefore \quad p = 0.2, q = 0.8$ and $n = 5$.

$\therefore \quad$ Probability of 2 successes $= P(2) = {}^5C_2 p^2 q^3$

$= 10(0.2)^2 (0.8)^3 = 0.2048$

Example 12. *If the sum and the product of the mean and variance of Binomial Distribution are 1.8 and 0.8 respectively, find the probability distribution and the probability of atleast one success.*

Solution. According to given, we have

$$np + npq = 1.8 \Rightarrow np(1+q) = \frac{9}{5} \qquad ...(i)$$

$$\text{and } np \,.\, npq = 0.8 \Rightarrow n^2p^2q = \frac{4}{5} \qquad ...(ii)$$

Dividing the square of (*i*) by (*ii*), we get

$$\frac{n^2p^2(1+q)^2}{n^2p^2q} = \left(\frac{9}{5}\right)^2 \times \frac{5}{4} \Rightarrow \frac{(1+q)^2}{q} = \frac{81}{20}$$

$$\Rightarrow \quad 20(1 + 2q + q^2) = 81q \Rightarrow 20q^2 - 41q + 20 = 0$$

$$\Rightarrow \quad (5q - 4)(4q - 5) = 0 \Rightarrow q = \frac{4}{5}, \frac{5}{4} \text{ but } 0 < q < 1$$

$$\Rightarrow \quad q = \frac{4}{5}.$$

$$\therefore \quad p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}.$$

From (*i*), $n \,.\, \frac{1}{5}\left(1 + \frac{4}{5}\right) = \frac{9}{5} \Rightarrow n = 5.$

Hence, the binomial distribution is $(q + p)^n$ *i.e.* $\left(\frac{4}{5} + \frac{1}{5}\right)^5$.

Probability of atleast one success $= 1 - P(0) = 1 - q^5$

$$= 1 - \left(\frac{4}{5}\right)^5 = 1 - \frac{1024}{3125} = \frac{2101}{3125}.$$

Example 13. *The difference between mean and variance of a binomial distribution is 1 and the difference of their squares is 11. Find the distribution.*

Solution. According to given, we have

$$np - npq = 1 \text{ i.e. } np(1-q) = 1 \qquad ...(i)$$

$$\text{and } (np)^2 - (npq)^2 = 11 \text{ i.e. } n^2p^2(1-q^2) = 11 \qquad ...(ii)$$

Dividing (*ii*) by the square of (*i*), we get

$$\frac{n^2p^2(1-q^2)}{n^2p^2(1-q)^2} = \frac{11}{1} \Rightarrow \frac{1-q^2}{(1-q)^2} = 11$$

$$\Rightarrow \quad \frac{(1-q)(1+q)}{(1-q)^2} = 11 \Rightarrow \frac{1+q}{1-q} = 11$$

$$\Rightarrow \quad 1+q = 11 - 11q \Rightarrow 12q = 10 \Rightarrow q = \frac{5}{6}.$$

$$\therefore \quad p = 1 - q = 1 - \frac{5}{6} = \frac{1}{6}.$$

From (*i*), we get

$$n \times \frac{1}{6}\left(1 - \frac{5}{6}\right) = 1 \Rightarrow n \times \frac{1}{6} \times \frac{1}{6} = 1 \Rightarrow n = 36.$$

$\therefore$ The binomial distribution is $\left(\frac{5}{6} + \frac{1}{6}\right)^{36}$.

Example 14. *A coin is tossed 400 times and it shows head 220 times. Discuss whether the coin is unbiased or not.*

Solution. Here $p = \frac{1}{2}, q = \frac{1}{2}, n = 400$. We see that n is fairly large.

Expected value (Mean) $= np = 400\left(\frac{1}{2}\right) = 200$

and standard deviation $= \sigma = \sqrt{npq} = \sqrt{400 . \frac{1}{2} . \frac{1}{2}} = 10.$

Observed value $220 = 200 + 2\,(10) = \text{Mean} + 2\sigma$.

Hence, the coin is *perhaps biased.*

Example 15. *If the mean of a binomial distribution is 81, then find the interval in which standard deviation of the binomial distribution lies.* *(CBSE 2022 Term I)*

Solution. Given mean $= np = 81$

$\therefore$ Standard deviation $= \sqrt{npq} = \sqrt{81q} = 9\sqrt{q}$

$\Rightarrow$ S.D. < 9 $\quad (\because q < 1)$

Also S.D. ≥ 0

$\Rightarrow$ $0 \leq$ S.D. < 9

$\therefore$ Standard deviation lies in the interval $[0, 9)$.

EXERCISE 9.4

1. Find the mean of the Binomial distribution $B\left(8, \frac{1}{4}\right)$.
2. In a binomial distribution, if $n = 20$ and $q = 0.6$, then find its mean.
3. In a binomial distribution, if mean is 5 and variance is 4, then find the number of trials.
4. If the mean of a binomial distribution is 20 and its standard deviation is 4, then find the value of p.
5. Find the expectation of number of heads in 20 tosses of a fair coin.
6. A die is tossed twice. A success is defined as getting an odd number on a random toss. Find the mean and variance of the number of successes.
7. If X follows binomial distribution with parameters $n = 5$, p and $P(X = 2) = 9\,P(X = 3)$, then find the value of p.

8. Three cards are drawn successively with replacement from a well-shuffled pack of 52 cards. Find the mean and the variance of the number of red cards drawn.

9. A die is thrown 3 times. Let X be 'the number of two's seen', then find the probability distribution of X. Also find the expectation of X.

10. Find the probability distribution of the number of successes in two tosses of a die, where a success is defined as 'a number greater than 4'. Also find the mean, variance and standard deviation of the distribution.

11. Two cards are drawn successively with replacement from a well-shuffled pack of 52 cards. Find the probability distribution of the number of aces drawn. Also find the mean and standard deviation of the distribution.

12. A die is tossed three times. Getting an even number is considered a success. Find the binomial distribution. Also find the variance of the distribution.

13. An urn contains 3 white and 6 red balls. Four balls are drawn one by one with replacement from the urn. Find the probability distribution of the number of red balls drawn. Hence, find the mean and variance of the distribution.

14. If X follows a binomial distribution with mean 3 and variance $\frac{3}{2}$, then find $P(X \le 5)$.

15. If the sum of the mean and the variance of a binomial distribution for 54 trials is 30, then find the distribution.

Answers

1. 2
2. 8
3. 25
4. $\frac{1}{5}$
5. 10
6. $1; \frac{1}{2}$
7. $\frac{1}{10}$
8. Mean $= \frac{3}{2}$, variance $= \frac{3}{4}$
9. $\begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{125}{216} & \frac{75}{216} & \frac{15}{216} & \frac{1}{216} \end{pmatrix}; \frac{1}{2}$
10. $\begin{pmatrix} 0 & 1 & 2 \\ \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \end{pmatrix}; \frac{2}{3}, \frac{4}{9}, \frac{2}{3}$
11. $\begin{pmatrix} 0 & 1 & 2 \\ \frac{144}{169} & \frac{24}{169} & \frac{1}{169} \end{pmatrix}; \frac{2}{13}, \frac{2}{13}\sqrt{6}$
12. $\left(\frac{1}{2} + \frac{1}{2}\right)^3; \frac{3}{4}$
13. $\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ \frac{1}{81} & \frac{8}{81} & \frac{24}{81} & \frac{32}{81} & \frac{16}{81} \end{pmatrix}; \frac{8}{3}, \frac{8}{9}$
14. $\frac{63}{64}$
15. $\left(\frac{2}{3} + \frac{1}{3}\right)^{54}$